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Induced representations of the group of diffeomorphisms of $\mathbb{R}^3$†

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Abstract. The class of representations of the diffeomorphism group introduced by Borisov (1979), based on SL(3, $\mathbb{R}$), is extended to a class based on the universal covering group $\overline{\mathrm{SL}(3, \mathbb{R})}$, and is generalised from the one-particle to the N-particle case. Earlier, the authors constructed induced representations of a local current group from little group representations that factor through the symmetric group S_N, which is a discrete group. The present class of representations is obtained through an analogous construction, first for the continuous group $\overline{\mathrm{SL}(3, \mathbb{R})}$, and then for the semi-direct product group $\mathrm{SL}(3, \mathbb{R})^N \wedge S_N$.

1. Introduction

The group of diffeomorphisms of $\mathbb{R}^3$ occurs naturally in the quantum theory of local observables. Let $\rho(f)$ denote the operator describing the averaged particle density at a fixed time, where f belongs to Schwartz's space $\mathcal{S}$ of C^∞ functions of rapid decrease at infinity. Let $J(\boldsymbol{g})$ similarly describe the particle flux, where $\boldsymbol{g}$ has components in $\mathcal{S}$. In a non-relativistic quantum theory, these operators satisfy the equal-time commutation relations

$$[\rho(f_1), \rho(f_2)] = 0, \tag{1}$$

$$[\rho(f), J(\boldsymbol{g})] = \mathrm{i}\rho(\boldsymbol{g}\cdot\nabla f), \tag{2}$$

$$[J(\boldsymbol{g}_1), J(\boldsymbol{g}_2)] = \mathrm{i}J([\boldsymbol{g}_1, \boldsymbol{g}_2]), \tag{3}$$

where $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is the Lie bracket $\boldsymbol{g}_2\cdot\nabla\boldsymbol{g}_1 - \boldsymbol{g}_1\cdot\nabla\boldsymbol{g}_2$ of the vector fields $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ on $\mathbb{R}^3$. Thus we have an infinite-dimensional Lie algebra (Dashen and Sharp 1968, Goldin and Sharp 1970, Grodnik and Sharp 1970).

The group associated with this algebra is a semi-direct product $\mathcal{S} \wedge \mathcal{K}$, where $\mathcal{S}$ is Schwartz's space taken under addition, and $\mathcal{K}$ is the group of C^∞ diffeomorphisms of $\mathbb{R}^3$ which (with all derivatives) tend rapidly toward the identity mapping at infinity,

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taken under composition (Goldin 1971). The group law is given by

$$(f_1, \boldsymbol{\phi}_1)(f_2, \boldsymbol{\phi}_2) = (f_1 + f_2 \circ \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \circ \boldsymbol{\phi}_1) \tag{4}$$

for $f_1, f_2 \in \mathcal{S}$ and $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{K}$.

A formalism for describing representations of nuclear spaces, described by Gel'fand and Vilenkin (1964), has been useful in studying representations of $\mathcal{S} \wedge \mathcal{K}$ (Goldin and Sharp 1970, Goldin 1971, Goldin *et al* 1974). A continuous unitary representation of $\mathcal{S}$ is described by a positive, normalised, countably additive measure μ on the σ-algebra of cylinder sets in $\mathcal{S}'$, the continuous dual of $\mathcal{S}$. To determine a continuous unitary representation of $\mathcal{S} \wedge \mathcal{K}$, μ must be quasi-invariant under the action of $\mathcal{K}$ in $\mathcal{S}'$. A class of irreducible representations of $\mathcal{S} \wedge \mathcal{K}$ is described by a quasi-invariant probability measure μ concentrated on a single $\mathcal{K}$-orbit Δ in $\mathcal{S}'$, together with a cocyle $\chi_{\boldsymbol{\phi}}(F)$ defined for $F \in \mathcal{S}'$ and $\boldsymbol{\phi} \in \mathcal{K}$.

The induced representation formalism, introduced by Mackey (1952) for semi-direct products of locally compact groups, has been generalised by the authors to the group $\mathcal{S} \wedge \mathcal{K}$ (Goldin *et al* 1980). The present paper places the class of representations given by Borisov (1979), based on SL(3, $\mathbb{R}$), in the framework of induced representations of $\mathcal{S} \wedge \mathcal{K}$. We also extend this class to include representations based on the covering group $\widetilde{\mathrm{SL}}(3, \mathbb{R})$. These representations are of order 1 in the sense described by Vershik *et al* (1975, p 48). The representations considered here correspond to the N-particle orbits

$$\Delta_N = \left\{ F \in \mathcal{S}' : F = \sum_{i=1}^{N} \delta_{\boldsymbol{x}_i}, \text{ with the } \boldsymbol{x}_i \text{ all distinct} \right\}, \tag{5}$$

where $\delta_{\boldsymbol{x}} \in \mathcal{S}'$ is the evaluation functional $(\delta_{\boldsymbol{x}}, f) = f(\boldsymbol{x})$. The action of $\boldsymbol{\phi} \in \mathcal{K}$ on $F = \Sigma_{i=1}^{N} \delta_{\boldsymbol{x}_i}$ is given by

$$\boldsymbol{\phi}^* F = \sum_{i=1}^{N} \delta_{\boldsymbol{\phi}(\boldsymbol{x}_i)}. \tag{6}$$

An important role is played by the little group

$$\mathcal{K}_{F_0} = \{ \boldsymbol{\phi} \in \mathcal{K} : \boldsymbol{\phi}^* F_0 = F_0 \}, \tag{7}$$

for F_0 a fixed element of Δ_N. In the inducing construction, the cocycle $\chi_{\boldsymbol{\phi}}(F)$ is determined from a representation of $\mathcal{K}_{F_0}$. Because $\mathcal{K}_{F_0}$ is not locally compact, our procedure for constructing induced representations has been to define a homomorphism from $\mathcal{K}_{F_0}$ to a locally compact group G, obtaining representations of $\mathcal{K}_{F_0}$ from those of G.

When G is taken to be the symmetric group S_N (the fundamental group of the orbit Δ_N), with the natural homomorphism from $\mathcal{K}_{F_0}$ onto S_N, a class of induced representations results which have been described by the authors (Goldin *et al* 1980). One-dimensional representations of S_N correspond to indistinguishable particles obeying Bose or Fermi statistics, while higher-dimensional representations correspond to particles obeying parastatistics (Dicke and Goldin 1983). Here the homomorphism from $\mathcal{K}_{F_0}$ to S_N depends only on the value of $\boldsymbol{\phi} \in \mathcal{K}_{F_0}$ at N points. More of the structure of each diffeomorphism can be utilised by defining homomorphisms which depend on derivatives of $\boldsymbol{\phi}$. The simplest such homomorphism in the case $N = 1$ maps $\mathcal{K}_{F_0}$ (where $F_0 = \delta_{\boldsymbol{x}_0}$) onto $\mathbb{R}^+$ (the positive reals under multiplication), by $\boldsymbol{\phi} \to \det \mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x}_0)$, with $[\mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x}_0)]_{jk} = \partial_j \phi_k(\boldsymbol{x}_0)$. Representations of $\mathcal{S} \wedge \mathcal{K}$ induced by representations of $\mathbb{R}^+$ were discussed elsewhere by the authors (Goldin *et al* 1981).

The representations introduced by Borisov correspond to the choices $N=1$, $G=\mathrm{SL}(3,\mathbb{R})$, and the homomorphism $\boldsymbol{\phi}\to[\det\mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x}_0)]^{-1/3}\mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x}_0)$. The representations of SL(3, $\mathbb{R}$) described by Sijacki (1975) can then be used to obtain representations of the group of diffeomorphisms of $\mathbb{R}^3$.

In § 2 of this paper, we use the condition that the diffeomorphism $\boldsymbol{\phi}\in\mathcal{K}$ tends toward the identity mapping at infinity to extend Borisov's representations to a class based on $\overline{\mathrm{SL}(3,\mathbb{R})}$. We then introduce needed notation, and describe these representations in the context of induced representations of $\mathcal{S}\wedge\mathcal{K}$. A physical interpretation with respect to particle spin will be described in another paper (Goldin and Sharp 1983). Borisov's class of representations is also extended in this section to the case $N>1$, where indistinguishable as well as distinguishable particles can be described. Concluding remarks are contained in § 3.

2. Induced representations based on $\overline{\mathrm{SL}(3,\mathbb{R})}$

We treat first the case of a one-particle orbit $\Delta_1=\{\delta_{\boldsymbol{x}}:\boldsymbol{x}\in\mathbb{R}^3\}$. For fixed $\boldsymbol{x}$, define the map $h_{\boldsymbol{x}}:\mathcal{K}\to\mathrm{SL}(3,\mathbb{R})$ by

$$[h_{\boldsymbol{x}}(\boldsymbol{\phi})]_{jk}=[\det\mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x})]^{-1/3}(\partial_j\phi_k)(\boldsymbol{x}). \tag{8}$$

Restricted to the little group $\mathcal{K}_{\delta_{\boldsymbol{x}}}=\{\boldsymbol{\phi}:\boldsymbol{\phi}(\boldsymbol{x})=\boldsymbol{x}\}$, $h_{\boldsymbol{x}}$ defines a homomorphism from $\mathcal{K}_{\delta_{\boldsymbol{x}}}$ onto SL(3, $\mathbb{R}$). To obtain a homomorphism from $\mathcal{K}_{\delta_{\boldsymbol{x}}}$ to $\overline{\mathrm{SL}(3,\mathbb{R})}$, we use the fact that for $\boldsymbol{\phi}\in\mathcal{K}$, $\boldsymbol{\phi}(\boldsymbol{x})$ approaches $\boldsymbol{x}$ as $|\boldsymbol{x}|\to\infty$, and its derivatives $\partial_j\phi_k(\boldsymbol{x})$ approach δ_{jk}. Consider a continuous path $\boldsymbol{x}_t$, $0<t\leq 1$, with $\lim_{t\to 0}|\boldsymbol{x}_t|=\infty$ and $\boldsymbol{x}_{t=1}=\boldsymbol{x}$. Then, for $0\leq t\leq 1$, $h_{\boldsymbol{x}_t}(\boldsymbol{\phi})$ defines a continuous path in SL(3, $\mathbb{R}$) from the identity element to $h_{\boldsymbol{x}}(\boldsymbol{\phi})$. Thus $h_{\boldsymbol{x}_t}(\boldsymbol{\phi})$ corresponds to an element of $\overline{\mathrm{SL}(3,\mathbb{R})}$. This element of $\overline{\mathrm{SL}(3,\mathbb{R})}$ is independent of the path $\boldsymbol{x}_t$ that is chosen, for if $\boldsymbol{x}_t$ and $\boldsymbol{x}'_t$ are two distinct paths from infinity to $\boldsymbol{x}$, then $h_{\boldsymbol{x}_t}(\boldsymbol{\phi})$ and $h_{\boldsymbol{x}'_t}(\boldsymbol{\phi})$ define distinct paths from the identity to $h_{\boldsymbol{x}}(\boldsymbol{\phi})$ in SL(3, $\mathbb{R}$). As we continuously deform the path $\boldsymbol{x}_t$ into $\boldsymbol{x}'_t$, the path $h_{\boldsymbol{x}_t}(\boldsymbol{\phi})$ is continuously deformed into $h_{\boldsymbol{x}'_t}(\boldsymbol{\phi})$, since all derivatives $\partial_j\phi_k$ vary continuously. Thus the paths in SL(3, $\mathbb{R}$) are homotopic, and define the same element of $\overline{\mathrm{SL}(3,\mathbb{R})}$. Call this element $\tilde{h}_{\boldsymbol{x}}(\boldsymbol{\phi})$.

Next we note that $\tilde{h}_{\boldsymbol{x}}$ satisfies the equation

$$\tilde{h}_{\boldsymbol{x}}(\boldsymbol{\phi}_1\circ\boldsymbol{\phi}_2)=\tilde{h}_{\boldsymbol{x}}(\boldsymbol{\phi}_2)\tilde{h}_{\boldsymbol{\phi}_2(\boldsymbol{x})}(\boldsymbol{\phi}_1). \tag{9}$$

This follows from the chain rule, $h_{\boldsymbol{x}_t}(\boldsymbol{\phi}_1\circ\boldsymbol{\phi}_2)=h_{\boldsymbol{x}_t}(\boldsymbol{\phi}_2)h_{\boldsymbol{\phi}_2(\boldsymbol{x}_t)}(\boldsymbol{\phi}_1)$, and the fact that $\boldsymbol{\phi}_2(\boldsymbol{x}_t)$ is a path in $\mathbb{R}^3$ from infinity to $\boldsymbol{\phi}_2(\boldsymbol{x})$. In particular, $\tilde{h}_{\boldsymbol{x}}$ restricted to $\mathcal{K}_{\delta_{\boldsymbol{x}}}$ is a homomorphism.

We next describe an inducing construction which defines the representation. First we introduce the fibre space $\mathcal{F}=\Delta\times\overline{\mathrm{SL}(3,\mathbb{R})}$ with the canonical projection $p:\mathcal{F}\to\Delta$. The action $\boldsymbol{\phi}^*$ of the diffeomorphism $\boldsymbol{\phi}$ on Δ lifts to $\mathcal{F}$ as follows:

$$\tilde{\boldsymbol{\phi}}^*(\delta_{\boldsymbol{x}},M)=(\boldsymbol{\delta}_{\boldsymbol{\phi}(\boldsymbol{x})},[\tilde{h}_{\boldsymbol{x}}(\boldsymbol{\phi})]^{-1}M), \tag{10}$$

where $M\in\overline{\mathrm{SL}(3,\mathbb{R})}$. This action satisfies the group law $\tilde{\boldsymbol{\phi}}_1^*\tilde{\boldsymbol{\phi}}_2^*=(\widetilde{\boldsymbol{\phi}_1\circ\boldsymbol{\phi}_2})^*$.

Consider a continuous unitary representation Π of $\overline{\mathrm{SL}(3,\mathbb{R})}$ in a Hilbert space $\mathcal{M}$. The class of such representations has been described by Sijacki (1975). Let $\mathcal{H}$ be the Hilbert space of functions $\tilde{\Psi}$ on $\mathcal{F}$, with values in $\mathcal{M}$, satisfying the following: (1) $\tilde{\Psi}$ is measurable with respect to the product measure $\tilde{\mu}=\mu\times\nu$, where μ is the given cylindrical measure concentrated on Δ_1, and ν is Haar measure on $\overline{\mathrm{SL}(3,\mathbb{R})}$; (2) $\tilde{\Psi}$

satisfies the symmetry condition

$$\tilde{\Psi}(\delta_x, MN) = \Pi(N^{-1})\tilde{\Psi}(\delta_x, M), \tag{11}$$

for all $M, N \in \overline{\mathrm{SL}(3,\mathbb{R})}$, so that the value of the inner product in $\mathcal{M}$, $(\tilde{\Psi}_1(\delta_x, M), \tilde{\Psi}_2(\delta_x, M))_{\mathcal{M}}$, depends only on x and is independent of M; (3) $\tilde{\Psi}$ is square integrable with respect to μ; i.e. $\int d\mu\ (\tilde{\Psi}, \tilde{\Psi})_{\mathcal{M}} < \infty$.

In $\mathcal{H}$, a representation of $\mathcal{S} \wedge \mathcal{K}$ is constructed as follows. Define $\mu_{\boldsymbol{\phi}}$ on Δ_1 by $\mu_{\boldsymbol{\phi}}(X) = \mu(\boldsymbol{\phi}^* X)$ for X a measurable subset of Δ_1. Then

$$[\mathcal{U}(f)\tilde{\Psi}](\delta_x, M) = \mathrm{e}^{\mathrm{i}f(x)}\tilde{\Psi}(\delta_x, M) \tag{12}$$

and

$$\begin{aligned}[\mathcal{V}(\boldsymbol{\phi})\tilde{\Psi}](\delta_x, M) &= \left[\frac{d\mu_{\boldsymbol{\phi}}}{d\mu}(\delta_x)\right]^{1/2} \tilde{\Psi}(\tilde{\boldsymbol{\phi}}^*(\delta_x, M)) \\ &= \left[\frac{d\mu_{\boldsymbol{\phi}}}{d\mu}(\delta_x)\right]^{1/2} \tilde{\Psi}(\delta_{\boldsymbol{\phi}(x)}, [\tilde{h}_x(\boldsymbol{\phi})]^{-1}M),\end{aligned} \tag{13}$$

where $d\mu_{\boldsymbol{\phi}}/d\mu$ is the Radon–Nikodym derivative. Defining $\Psi(\delta_x) = \tilde{\Psi}(\delta_x, E)$, where E is the identity element in $\overline{\mathrm{SL}(3,\mathbb{R})}$, we have

$$[\mathcal{V}(\boldsymbol{\phi})\Psi](\delta_x) = \left[\frac{d\mu_{\boldsymbol{\phi}}}{d\mu}(\delta_x)\right]^{1/2} \Pi(\tilde{h}_x(\boldsymbol{\phi}))\Psi(\delta_{\boldsymbol{\phi}(x)}). \tag{14}$$

Thus we have obtained the Gel'fand–Vilenkin form of the representation, in which the cocycle is given by $\chi_{\boldsymbol{\phi}}(\delta_x) = \Pi(\tilde{h}_x(\boldsymbol{\phi}))$, with $\Psi \in \mathcal{L}^2_\mu(\Delta, \mathcal{M})$.

Now let us look at the infinitesimal generators of such a representation. First suppose that Π is actually a representation of $\mathrm{SL}(3,\mathbb{R})$ in $\mathcal{M}$. Let $J_0(\boldsymbol{g})$ be the operator $(2\mathrm{i})^{-1}(\boldsymbol{g}\cdot\nabla + \nabla\cdot\boldsymbol{g})$ acting on the spatial coordinates of $\Psi \in \mathcal{H}$, where $\Psi(\delta_x)$ is now considered as a square-integrable function of x taking values in $\mathcal{M}$. Let Σ_-, Σ_0, Σ_+ be the generators of SO(3) considered as a subgroup of $\mathrm{SL}(3,\mathbb{R})$, and T_{-2}, T_{-1}, T_0, T_1, T_2 be the quadrupole operators. These generators are defined by Sijacki (1975) in a spherical basis as follows:

$$\Sigma_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\mathrm{i} \\ -1 & \mathrm{i} & 0 \end{pmatrix}, \qquad \Sigma_0 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Sigma_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\mathrm{i} \\ 1 & \mathrm{i} & 0 \end{pmatrix}, \tag{15}$$

$$T_0 = -\mathrm{i}\sqrt{\tfrac{2}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad T_{\pm1} = \begin{pmatrix} 0 & 0 & \mp\mathrm{i} \\ 0 & 0 & 1 \\ \mp\mathrm{i} & 1 & 0 \end{pmatrix}, \qquad T_{\pm2} = \begin{pmatrix} \mathrm{i} & \mp1 & 0 \\ \mp1 & -\mathrm{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

and satisfy the commutation relations

$$\begin{aligned} &[\Sigma_0, \Sigma_\pm] = \pm\Sigma_\pm, \qquad [\Sigma_0, T_\mu] = \mu T_\mu (\mu = 0, \pm1, \pm2), \\ &[\Sigma_+, \Sigma_-] = 2\Sigma_0, \qquad [\Sigma_\pm, T_\mu] = [6 - \mu(\mu \pm 1)]^{1/2} T_{\mu\pm1}, \qquad [T_2, T_{-2}] = -4\Sigma_0. \end{aligned} \tag{17}$$

With $\Sigma_x = \frac{1}{2}(\Sigma_+ + \Sigma_-)$, $\Sigma_y = (2\mathrm{i})^{-1}(\Sigma_+ - \Sigma_-)$ and $\Sigma_z = \Sigma_0$, we write

$$J(\boldsymbol{g}) = J_0(\boldsymbol{g})\cdot I + \tfrac{1}{2}(\nabla\times\boldsymbol{g})\cdot\boldsymbol{\Sigma} + \tfrac{1}{2}G_{-\mu}(\boldsymbol{g})T_\mu \tag{18}$$

where

$$G_0(\boldsymbol{g}) = \sqrt{\tfrac{1}{6}}(\partial_1 g^1 + \partial_2 g^2 - 2\partial_3 g^3),$$
$$G_1(\boldsymbol{g}) = \overline{-G_{-1}(\boldsymbol{g})} = -\tfrac{1}{2}[(\partial_1 g^3 + \partial_3 g^1) + \mathrm{i}(\partial_2 g^3 + \partial_3 g^2)], \tag{19}$$
$$G_2(\boldsymbol{g}) = \overline{G_{-2}(\boldsymbol{g})} = -\tfrac{1}{2}[\partial_1 g^1 - \partial_2 g^2 + \mathrm{i}(\partial_1 g^2 + \partial_2 g^1)],$$

and where summation over μ is understood.

In a unitary representation of SL(3, $\mathbb{R}$), Σ_x, Σ_y and Σ_z are represented by self-adjoint operators in $\mathcal{M}$, and the T_μ satisfy $T_\mu^* = (-1)^\mu T_{-\mu}$. In (18), each term is a tensor product of an operator on spatial coordinates with an operator in $\mathcal{M}$. Now $\rho(f)$ and $J(\boldsymbol{g})$ define a representation of the algebra in (1)–(3), as can be verified by a straightforward computation. For a representation of the covering group $\overline{\mathrm{SL}(3,\mathbb{R})}$, equation (18) continues to hold, but the operators $\boldsymbol{\Sigma}$ now represent generators of SU(2), the maximal compact subgroup of $\overline{\mathrm{SL}(3,\mathbb{R})}$.

In representations of $\mathcal{S} \wedge \mathcal{K}$ previously described (Goldin *et al* 1981) we map $\mathcal{K}_{\delta_x} \to \mathbb{R}^+$ by $\boldsymbol{\phi} \to \det \mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x})$ and consider representations of $\mathbb{R}^+$ given by $r \to \mathrm{e}^{\mathrm{i}\lambda \ln r}$ for fixed λ. Thus we obtain a unitary representation of $\mathcal{K}_{\delta_x}$ by the map $\boldsymbol{\phi} \to \exp[\mathrm{i}\lambda \ln \det \mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x})]$, which induces a representation of $\mathcal{S} \wedge \mathcal{K}$. In this representation, $\chi_{\boldsymbol{\phi}}(\delta_x) = \exp[\mathrm{i}\lambda \ln \det \mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x})]$ and $J(\boldsymbol{g}) = J_0(\boldsymbol{g}) + \lambda\rho(\nabla\cdot\boldsymbol{g})$. The above two classes of representations are easily combined by taking a new cocycle $\chi_{\boldsymbol{\phi}}(\delta_x)$ to be the product of the preceding two. This is possible because we can define a homomorphism from the little group $\mathcal{K}_{\delta_x}$ to $\overline{\mathrm{GL}(3,\mathbb{R})}$, where $\overline{\mathrm{GL}(3,\mathbb{R})}$ is the product of $\mathbb{R}^+$ with $\overline{\mathrm{SL}(3,\mathbb{R})}$. Thus we have in general:

$$J(\boldsymbol{g}) = J_0(\boldsymbol{g}) + \lambda\rho(\nabla\cdot\boldsymbol{g}) + \tfrac{1}{2}(\nabla\times\boldsymbol{g})\cdot\boldsymbol{\Sigma} + \tfrac{1}{2}G_{-\mu}(\boldsymbol{g})T_\mu. \tag{20}$$

Next let us consider the case of the N-particle orbit. The little group $\mathcal{K}_F$ for $F = \sum_{j=1}^N \delta_{x_j}$ contains those diffeomorphisms $\boldsymbol{\phi}$ which permute the points $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$; i.e. for which the set $\{\boldsymbol{\phi}(\boldsymbol{x}_1), \ldots, \boldsymbol{\phi}(\boldsymbol{x}_N)\}$ equals the set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$. Suppose that $\boldsymbol{\phi}(\boldsymbol{x}_j) = \boldsymbol{x}_k$; then the Jacobian matrix $\mathcal{J}_{\boldsymbol{\phi}}(\boldsymbol{x}_j)$ defines a linear mapping from $T_{x_j}(\mathbb{R}^3)$ to $T_{x_k}(\mathbb{R}^3)$, where $T_x(\mathbb{R}^3)$ denotes the tangent space to $\mathbb{R}^3$ at $\boldsymbol{x}$. The tangent space to the orbit Δ_N at F, denoted by $T_F(\Delta_N)$, is a $3N$-dimensional space which as a vector space is the direct sum of the $T_{x_j}(\mathbb{R}^3)$. Now for any $\boldsymbol{\phi}$ in the little group $\mathcal{K}_F$, there is a unique linear transformation defined by $[\det \mathcal{J}_{\boldsymbol{\phi}}]^{-1/3}\mathcal{J}_{\boldsymbol{\phi}}$ from $T_F(\Delta_N)$ to itself, whose restriction to $T_{x_j}(\mathbb{R}^3)$ is $T_{x_k}(\mathbb{R}^3)$, regarding $T_{x_j}(\mathbb{R}^3)$ and $T_{x_k}(\mathbb{R}^3)$ as subspaces of $T_F(\Delta_N)$. The set of all such transformations forms a group L, namely the subgroup of SL($3N$, $\mathbb{R}$) which acts on $\mathbb{R}^{3N}$ so as to preserve the given family of three-dimensional subspaces. Furthermore the map defined above is a homomorphism h_F from $\mathcal{K}_F$ to L.

Now $\mathrm{SL}(3,\mathbb{R})^N$ is a normal subgroup of L, consisting of those transformations of $\mathbb{R}^{3N}$ preserving the individual subspaces; and the quotient group $\mathrm{L}/\mathrm{SL}(3,\mathbb{R})^N$ is isomorphic to the symmetric group S_N. We can define an action of elements of S_N on $\mathrm{SL}(3,\mathbb{R})^N$, but this presupposes a choice of corresponding bases in $T_{x_i}(\mathbb{R}^3)$ for $i = 1, \ldots, N$, and there is no canonical way to make such a choice. Assuming a choice of bases, L may be written as the semi-direct product $\mathrm{SL}(3,\mathbb{R})^N \wedge \mathrm{S}_N$, and we can apply the usual Mackey theory for locally compact groups to consider its representations as extensions of representations of $\mathrm{SL}(3,\mathbb{R})^N$. Then, from each representation of L, a representation of $\mathcal{S} \wedge \mathcal{K}$ can be induced.

Let $(M_1, \ldots, M_N) \in \mathrm{SL}(3,\mathbb{R})^N$, and let Π_j be a unitary representation of SL(3, $\mathbb{R}$) acting in the Hilbert space $\mathcal{M}_j$, for $j = 1, \ldots, N$. A unitary representation Π of $\mathrm{SL}(3,\mathbb{R})^N$

is given by $\Pi(M_1, \ldots, M_N) = \Pi_1(M_1) \otimes \ldots \otimes \Pi_N(M_N)$, acting in the Hilbert space $\otimes_{j=1}^{N} \mathcal{M}_j$.

Suppose the representations Π_j are all the same. Then, in the representation Π, the little group in S_N of the element $(M_1, \ldots, M_N)$ is the full group S_N itself. Thus any representation of S_N induces a representation of L. The measure for this representation is concentrated on an orbit which may be thought of as describing N indistinguishable particles, while the 'internal symmetry' of each particle is described by the same representation Π_j of $SL(3, \mathbb{R})$. The choice of representation of S_N describes the particle statistics as has been previously explained (Goldin *et al* 1980).

At the other extreme, suppose the representations Π_j are all distinct. Then the little group in S_N of the element $(M_1, \ldots, M_N)$ in the representation Π is just $\{e\}$ in S_N. In this case, the induced representation of L is unique. Each particle has a distinct 'internal symmetry' described by the Π_j. Thus the particles are distinguishable, with no new representations arising from S_N to characterise exchange symmetry.

Similarly we can define a homomorphism from $\mathcal{K}_F$ to $\overline{SL(3, \mathbb{R})}^N \wedge S_N$, and the above remarks apply.

3. Conclusions

This paper continues the study of induced representations of the group of diffeomorphisms arising in quantum theory. In earlier work, we considered representations of the little group for the N-particle orbit which factor through S_N, and we obtained induced representations in the Hilbert space of wavefunctions on the covering space of the orbit satisfying a (Bose or Fermi) symmetry condition. Those representations utilised information only about the values of diffeomorphisms at N points. We have seen that a larger family of representations is obtained by including more of the structure of the diffeomorphisms. By considering representations of the same little group which factor through $SL(3, \mathbb{R})^N \wedge S_N$, we can include information about the Jacobian matrices of the diffeomorphisms at N points in the representation. Moving from $SL(3, \mathbb{R})$ to $\overline{SL(3, \mathbb{R})}$, information about the Jacobian matrices on paths extending to infinity is also included. The induced representations in these cases act in the space of wavefunctions on a fibre bundle over the orbit. Now the fibre is the continuous group $SL(3, \mathbb{R})$ or $\overline{SL(3, \mathbb{R})}$ or, more generally, $SL(3, \mathbb{R})^N \wedge S_N$, rather than the discrete group S_N considered previously.

A key role in the inducing construction is played by the lifting of the action $\boldsymbol{\phi}^*$ of a diffeomorphism from the orbit to the fibre bundle over the orbit. For the case at hand, we have explicitly constructed a lifting with the appropriate properties. This example strongly suggests further generalisation of the inducing construction for non-locally compact groups described earlier (Goldin *et al* 1980), from the case where the representation of the little group factors through the fundamental group of an orbit, to the case where the representation of the little group factors through any locally compact group.

In summary, when the Gel'fand–Vilenkin measure is concentrated on a one-particle orbit, we have recovered in the induced representation formalism the class of representations obtained by Borisov (1979), and we have extended the class to representations based on $\overline{SL(3, \mathbb{R})}$. We have also obtained induced representations for the case when the Gel'fand–Vilenkin measure is concentrated on an N-particle orbit, finding a natural unification of Borisov's representations with those previously obtained to describe particle statistics. Thus, the induced representation formalism provides a useful approach to studying representations of the non-locally compact group $\mathcal{S} \wedge \mathcal{K}$.

In closing, we observe that further classes of representations of the diffeomorphism group can be obtained by considering homomorphisms which depend on higher derivatives of the diffeomorphisms.

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